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Arabes, dans leurs travaux sur ces matières, font preuve vis-à-vis de leurs devanciers grecs.

Délai fixé: *31 octobre, 1889.* Langues: *latin, français, anglais, allemand, suédois, danois.* Adresser les Mémoires (portant une devise et accompagnés d'un billet cacheté muni de la même devise, et renfermant le nom, la profession, et l'adresse de l'Auteur) au Secrétaire de l'Académie, M. *H.-G. Zeuthen* professeur à l'Université de Copenhague.

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TEIXEIRA'S INFINITESIMAL ANALYSIS.*

One fourth of this work is occupied by an introduction containing two chapters: one on imaginaries, including their geometrical interpretation; the other on the general theory of functions. The remainder of the work, including about 250 pages, is devoted exclusively to the differential calculus. The method employed is that of infinitesimals, founded upon the method of limits. The fundamental principles are in general clearly stated, and nearly every page gives evidence that the author is well acquainted with the latest results. The work closes with chapters on functions defined by series, singularities of functions, and functions of imaginary variables. [*O. S.*]

*CURSO DE ANALYSE INFINITESIMAL por F. Gomes Teixeira. Porto: Typographia Occidental.

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SOLUTIONS OF EXERCISES.

132

A BROKEN line *ABCDE*, . . . etc., is drawn in a plane, having all its angles equal and the concavity always on the same side. Each of the successive parts *BC*, *CD*, *DE*, etc., is half as long as the preceding. The length and direction of *AB* are given and the common angle. Required the direction and distance from *A* of the point to which the end of the line approaches, when the construction as described is continued indefinitely. [*Yale Problems.*]

GENERALIZATION.

Let α be the vector *AB*; m the ratio of the length of any line to the length of the next; $\pi - \theta$ the common angle; ρ the vector from A to the required point, and λ the angle between ρ and α; m being greater than unity.

We shall have

$$\rho = \alpha + m^{-1}(\cos\theta + \varepsilon\sin\theta)\,\alpha + m^{-2}(\cos\theta + \varepsilon\sin\theta)^2\,\alpha + \ldots .$$

Whence
$$(1 - \cos\theta + \varepsilon\sin\theta)\,m\,\rho = \alpha.$$

If it be desired to express ρ in terms of θ, m, and $T\alpha\,(=a)$, we have, by squaring both members and reducing,

$$T\rho = \frac{ma}{\sqrt{(1 + m^2 - 2m\cos\theta)}}.$$

Acting on the same equation with $V(\alpha x)$ we have, after reduction,

$$\tan\lambda = \frac{\sin\theta}{m - \cos\theta}.$$ [*J. N. James.*]

139

RESTATED.

An ellipse touches the given ellipse $a^2y^2 + b^2x^2 - a^2b^2$ at the extremities of a diameter and has its foci on the curve. Find the position of its axis major; and the position of the foci when the angle between the diameter and the axis major of the greater ellipse is a maximum. [*R. H. Graves.*]

SOLUTION.

It is known that, "if any two chords OP, OQ of a conic make equal angles with the tangent at O, the line PQ will cut that tangent in a fixed point." (See Smith's Conic Sections, p. 211).

The fixed point is clearly the pole of the normal at O.

It follows that the required foci are the points of intersection of the given ellipse and a right line joining its centre and the pole of the normal at $O\,(x',y')$.

The equation to this line is

$$\frac{xx'}{a^4} + \frac{yy'}{b^4} = 0. \tag{1}$$

The chord of contact and (1) are conjugate diameters of the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{k^2}. \tag{2}$$

The angle between them is a maximum when they are equi-conjugate diameters of (2); then,

$$x' = \frac{a^2}{\sqrt{(a^2 + b^2)}} \quad \text{and} \quad y' = \frac{b^2}{\sqrt{(a^2 + b^2)}}.$$

Or, the foci and the points ($\pm x'$, $\pm y'$) are the points of intersection of

$$(a^2 + b^2)(x^2 + y^2) = a^4 + b^4,$$

and

$$b^2x^2 + a^2y^2 - a^2b^2 = 0.$$

[*R. H. Graves.*]

143

IF a curve of the fourth order have a singular point at which are three coincident tangents, this line passes through the intersection point of the bitangent with the line of junction of the two points of inflexion of the curve. [*F. H. Loud.*]

SOLUTION.

Writing the equation of the curve

$$4l^2y^3 = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

and putting $u = y/x$, I find, for the points of inflexion,

$$a + 2bu + cu^2 = 0.$$

The co-ordinates x and y are easily expressed in terms of u, and hence the equation

$$x(y' - y'') - y(x' - x'') + x'y'' - x''y' = 0$$

of a line through two given points may be written in terms of the parameters u_1 and u_2 of the two points, and becomes

$$\begin{aligned} x\left[a(u_1^4 - u_2^4) + 4bu_1u_2(u_1^3 - u_2^3) + 6cu_1^2u_2^2(u_1^2 - u_2^2) + 4du_1^3u_2^3(u_1 - u_2)\right] \\ - y\left[a(u_1^3 - u_2^3) + 4bu_1u_2(u_1^2 - u_2^2) + 6cu_1^2u_2^2(u_1 - u_2) - eu_1^3u_2^3(u_1 - u_2)\right] \\ - 4l^2u_1^3u_2^3(u_1 - u_2) = 0, \end{aligned}$$

and when u_1 and u_2 are the roots of the equation, $a + 2bu + cu^2 = 0$, the above becomes

$$(8b^3 - 12abc + 4a^2d)x + (4b^2c - 5ac^2 + a^2e)y - 4a^2l^2 = 0.$$

Also the equation of the curve is written without difficulty in the form

$$\begin{aligned} y^3\{(12a^2bc - 8ab^3 - 4a^3d)x + (9a^2c^2 - 12ab^2c + 4b^4 - a^3e)y + 4a^3l^2\} \\ = [a^2x^2 + 2abxy + (3ac - 2b^2)y^2]^2. \end{aligned}$$

Here the expression in $\{\}$, put equal to zero, is the equation of the bitangent, and comparing this equation with that of the line through the inflexions, it is readily seen that the two lines meet at a point of the axis of x, the singular tangent.

[*F. H. Loud.*]

157

FIND the locus of the instantaneous centre of a tangent to a parabola when one point of the tangent moves in the tangent at the vertex.

[*De Volson Wood.*]

SOLUTION.

The instantaneous centre is the join of the normal and the parallel to the axis from the point where the tangent cuts the tangent at the vertex. This point is the middle point of the normal. If the curve be $y^2 = 2px$, the normal at (x', y') is

$$y - y' = -\frac{y'}{p}(x - x'),$$

and its extremities are (x', y') and $(x' + p, 0)$. Its middle point is

$$x = x' + \tfrac{1}{2}p, \quad y = \tfrac{1}{2}y';$$

accordingly the required locus is

$$4y^2 = 2px - p^2,$$

a parabola whose parameter is $\frac{1}{4}p$ and whose vertex is at the focus of the original curve. [*W. M. Thornton.*]

159 and 160

FIND the centre of gravity of the loop of the Folium of Descartes.

SHOW that the circles of curvature at the node of the Folium of Descartes pass through the middle point of the arc of the loop. [*R. H. Graves.*]

SOLUTION.

The equation of the Folium of Descartes in Cartesian co-ordinates is

$$y^3 - 3axy + x^3 = 0.$$

The form of the equation shows that the curve is symmetrical with respect to an axis which bisects the angle between the original axes. If we transform to this by putting $(x - y)\sqrt{\frac{1}{2}}$ and $(x + y)\sqrt{\frac{1}{2}}$ in place of x and y, we obtain

$$y^2(b + 3x) = x^2(b - x);$$

$$\therefore y = \pm x \left\{\frac{b - x}{b + 3x}\right\}^{\frac{1}{2}},$$

where $$b = \tfrac{3}{2}\sqrt{2} \, . \, a.$$

The ordinate y having values with opposite signs between $x = 0$ and $x = b$, it follows that the centre of gravity of the loop is on the axis of x.

Let dS = an element of the loop's area, and x_1 = the abscissa of the centre of gravity. Hence

$$x_1 = \frac{\int_0^b x\,dS}{\int_0^b dS},$$

and $dS = 2y\,dx = 2x\left(\frac{b-x}{b+3x}\right)^{\frac{1}{2}}dx$ using the positive value of y. Hence

$$x_1 = \frac{\int_0^b x^2\left(\frac{b-x}{b+3x}\right)^{\frac{1}{2}}dx}{\int_0^b x\left(\frac{b-x}{b+3x}\right)^{\frac{1}{2}}dx}.$$

Let $\left(\frac{b-x}{b+3x}\right)^{\frac{1}{2}} = p$, then x_1 is readily transformed to

$$x_1 = b\frac{\int_1^0 \frac{p^2(1-p^2)^2}{(1+3p^2)^4}dp}{\int_1^0 \frac{p^2(1-p^2)}{(1+3p^2)^3}dp}.$$

But $$\int\frac{p^2(1-p^2)^2}{(1+3p^2)^4}dp = \frac{-p+10p^3-9p^5}{54(1+3p^2)^3} + \frac{1}{54\sqrt{3}}\tan^{-1}(p\sqrt{3}),$$

and $$\int\frac{p^2(1-p^2)}{(1+3p^2)^3}dp = \frac{\frac{1}{3}p^3}{(1+3p^2)^2};$$

$$\therefore\ x_1 = \frac{8\pi b}{27\sqrt{3}} = \frac{8\pi}{9\sqrt{6}}a.$$

To find the radius of curvature of the upper branch of the loop of the node, we have from the equation of the curve

$$y^2 = \frac{x^2(b-x)}{b+3x},$$

and the radius of curvature is

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}};$$

$$\therefore\ \rho = -\frac{(2b^4+8b^3x+12b^2x^3-18x^4)^{\frac{3}{2}}}{4b^3(b+3x)^2}.$$

At the node, when $x = 0$, $\qquad \rho_0 = -b\sqrt{\tfrac{1}{2}};$

the negative sign denoting that ρ_0 is counted from the node obliquely *downward* for the upper branch of the loop. This being understood, we may disregard the sign, and write for its length merely $\rho_0 = b\sqrt{\tfrac{1}{2}}$.

We have also $$\frac{dy}{dx} = \frac{b^2 - 3x^2}{(b - x)^{\frac{1}{2}}(b + 3x)^{\frac{3}{2}}};$$

hence at the node $$\frac{dy}{dx} = 1;$$

$$\therefore \tan \varphi_0 = 1;$$
$$\therefore \varphi_0 = \tfrac{1}{4}\pi.$$

But the radius ρ_0 is perpendicular to the tangent, and hence makes with the axis of x the angle $-\frac{1}{4}\pi$. If (α, β) are co-ordinates of the centre of curvature, then

$$\alpha = \rho_0 \cos(-\tfrac{1}{4}\pi) = \tfrac{1}{2}b,$$
$$\beta = \rho \sin(-\tfrac{1}{4}\pi) = -\tfrac{1}{2}b.$$

Hence the equation of the circle of curvature for the upper branch of the loop is

$$(x - \tfrac{1}{2}b)^2 + (y + \tfrac{1}{2}b)^2 = \tfrac{1}{2}b^2,$$

or $$x^2 + y^2 + b(y - x) = 0.$$

If this equation be combined with that of the loop, the abscissa of the intersections will be determined by the condition

$$x(x - b) = 0;$$

$\therefore x = 0$, the node for one point,

$x = b$, the abscissa of the middle of the whole arc of the loop.

The radius of curvature for the lower branch of the loop has the same value with the opposite sign, so that the centre of curvature is on the opposite side of the symmetrical axis. It is evident that it will pass through the same points,

$$(x = 0, \quad y = 0), \quad (x = b, \quad y = 0).$$

The area of the loop is $S = \frac{1}{3}b^2$. [*George W. Coakley.*]

163

An elliptic plate of semiaxes a and b, thickness c, and density δ, is revolved at an angular velocity Ω, about a line parallel to its minor axis, and at a fixed distance from it, in a plane through this axis perpendicular to the plane of the plate. Find the straining action on the plate along the line of its minor axis; and the greatest safe value of Ω, if σ be the admissible stress for unit of area on the material of the plate. [*Jas. S. Miller.*]

SOLUTION.

Consider an element of volume of the plate given by the relation

$$dV = y\,dx\,dz.$$

Taking the normal component of the centrifugal force exerted by this elementary mass, we have for the elementary moment about the axis of the plate

$$dM = \delta \Omega^2 \frac{b}{a} (a^2 - x^2)^{\frac{1}{2}} xz dx dz,$$

where z is the distance from the rotation axis to the minor axis of the elliptic lamina. Hence if d be the distance from the fixed axis to the centre line of the plate, we have for the total bending moment

$$M = 2\delta \Omega^2 \frac{b}{a} \int_{d - \frac{1}{2}c}^{d + \frac{1}{2}c} \int_{0}^{a} (a^2 - x^2)^{\frac{1}{2}} xz dx dz$$

$$= \tfrac{2}{3} \delta \Omega^2 b a^2 dc.$$

In a similar manner we have for the total stress produced by the sum of the components parallel to the plane of the plate,

$$P = 2\delta \Omega^2 \frac{b}{a} \int_{d - \frac{1}{2}c}^{d + \frac{1}{2}c} \int_{0}^{a} (a^2 - x^2)^{\frac{1}{2}} x dx dz$$

$$= \tfrac{2}{3} \delta \Omega^2 b a^2 c.$$

Hence, since the moment of inertia of the rectangular section is given by

$$I = \tfrac{1}{6} bc^3,$$

we have for the maximum intensity of stress, in absolute units,

$$\sigma = \frac{\delta \Omega^2 a^2}{3c} (c + 6d).$$

If σ be given, then obviously

$$\Omega = \sqrt{\left\{ \frac{3c\sigma}{\delta a^2 (c + 6d)} \right\}}.$$

[*Jas. S. Miller.*]

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EXERCISES.

195

UNDER what conditions will the equation

$$(a - \tfrac{1}{2}) x^3 + \tfrac{1}{2} x^2 + x - (1 + x) \log (1 + x) = 0$$

have positive real roots? [*R. S. Woodward.*]